SOME FIXED POINT THEOREMS IN L-SPACE

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Abstract: *In this paper the concept of L-SPACE is introduced and common fixed point theorems proved for commuting mappings in L-space.*

Keywords: L-space, Commuting Mapping, Fixed point, Common Fixed point.

INTRODUCTION

L-Space was introduced in fixed point theory by S. Kasahara in 1976.It is observed that in many fixed point theorems the metric properties ,in particular the axioms of triangle property ,are not essential in their proofs.

Let N denote the set of all non- negative integer. A pair (x →) of a set X and a subset → of the set X^N XX is called an L-space if the following two conditions are satisfied:

1. If $x_n=x$ ƐX for all nƐ N, then ({x_n }n ƐN, x) Ɛ→ **2.** If ({x_n }n ƐN, x) Ɛ→,then ({x_{ni}} iƐ N, x) Ɛ→For every subsequence {x_{ni}} I Ɛ N →of {x_n} n ƐN We shall write {x_n} n ƐN→ or {x_n} →x, instead of {x_n} n ƐN, X Ɛ→, and read {x_n} n ƐN converges to x.We require some definitions:

DEFINITION 1: Let (X, →) be an L-space. It is said to be separated if each sequence in X converges to at most one point of X.

DEFINITION 2: A mappings f of X into L-space (X', →) is said to be continuous if x_n→x implies f (x_{ni})→'f(x) for some subsequence {x_{ni}} i ƐN of {x_n} n ƐN.

DEFINITION 3: Let d be a non-negative extended real valued function on XxX: 0≤d(x, y) ⩽∞ for all x, y in X.The L-space(X, →) is said to be complete if each sequence {x_n} n ƐN in X with

$$\sum_{i=0}^{\infty} d(xi, xi+1)<\infty$$

Converges to at most one point at X.

In this context Kasahara S. proved a Lemma.

RELATED WORK

LEMMA: (K.S.) Let (X, →) be an L-space which is d-complete for a non-negative real valued function d on Xx X, if (X, →) is separated, then d(x ,y) = d(y ,x)=0, implies x=y for every x ,y in X.During the past few years many great mathematicians worked on L-space, respectively their names are Kasahara S.(1,2),Iseki(3),Singh(4).More recently, Pachpatte,B.G.(5,6) and Pathak H.K. and Dubey R.P.(7,8) gave very interesting results in L-space.In 1988, Pathak and Dubey proved theorem.

THEOREM: (P.D)- Let (X, →) be a separated L-space which is d-complete for a non-negative real valued function d on X xX with d(x, x)=0 for each x in X. Let E, F and T be three continuous self mappings of X satisfying the conditions.

1. ET=TE, FT=TF , F(X)C T(X) and E(X) C T(X) and

2 .d (Ex, Fy) ≤α $\frac{d(Tx,Ex)d(Ty,Fy)}{d(Tx,Ty)+\beta.d(Tx,Ty)}$

For all x, y in X where α,β≥0 and α+β <1 with T x ≠ Ty.

Then E, F and T have a common fixed point. Further, if α, β ≥0 and α+β <1 then E, F and T have a unique common fixed point. Further, Sharma and

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Agrawal proved the following theorem in their paper (9).

THEOREM: (S.A.) - Let (X, →) be a separated L-space which is d-complete for a non-negative real valued function d on XxX with d(x,x)=0 for each x in X. Let E, F and T be three continuous self mappings of X satisfying the conditions.

1. ET=TE, FT=TF,E(X)CT(X) and F(X) C T(X)

2. $d(Ex,Fy) \leq \alpha . \frac{d(Ty,Fy)[1+d(TxEx)]}{1+d(Tx,Ty)} + \beta . d(Tx,Ty)$

For all x, y in X where α,β≥0 and α+β <1 with T x ≠ Ty.

Then E, F and T have a common fixed point. Further, if α, β ≥0 and α+β <1 then E, F and T have a unique common fixed point. Our object in this paper is to extend and generalize the results of Jungck (9), Pathak and Dubey (6), Sharma and Agrawal (8) in fact we prove-

MAIN RESULT

THEOREM 1: Let (X, →) be a separated L-space which is d-complete for a non-negative real valued function d on Xx X with d(x, x) =0 for each x in X. Let E, F and T be three continuous self mappings of X satisfying the following conditions.

1.1. ET=TE,FT=TF,E(X)CT(X) and

F(X) C T(X)

1.2. $d(Ex,Fy) \leq \alpha \frac{d(Ty,Fy)[1+d(Tx,Ex)]}{1+d(Tx,Ty)}$

$+ \beta . \frac{d(Tx,Ex) d(Ty,Fy)}{d(Tx,Ty)} + \gamma [d(Tx,Ex) + d(Ty,Fy)] + \delta d(Tx,Ty)$

For all x, y in X where α, β, γ, δ ≥0 and α+β+2 γ +δ <1 with Tx ≠ Ty. Then E, F and T have a common fixed point.

Let x_0 Ɛ X. Since E(X)CT(X).We can choose a point x, in X such that $Tx_1=Ex_0$ also F(X) C T(X).we can choose a point x_2 in X such that $Tx_2=Fx_1$. In general we can choose the points.

$Tx_{2n+1}=Ex_{2n}$ …………… (a)

$Tx_{2n+2} =Fx_{2n+1}$ ………… (b)

Now we consider, $d(Tx_{2n+1}, Tx_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})$

$\leq \alpha \frac{d(Tx2n+1,Fx2n+1)[1+d(Tx2n,Ex2n)]}{1+d(Tx2n,Tx2n+1)}$

$+ \beta \frac{[d(Tx2n+1,Fx2n+1,)[d(Tx2n,Ex2n)]}{d(Tx2n,Tx2n+1)}$

$+ \gamma [d(Tx2n, Ex2n) + d(Tx2n+1, Fx2n+1)] + \delta d(Tx2n, T2n+1)$

$= \alpha \frac{d(Tx2n+1,Tx2n+2,)[1+d(Tx2n,Tx2n+1)]}{1+d(Tx2n,Tx2n+1)}$

$+ \beta \frac{d(Tx2n+1,Tx2n+2,)[d(Tx2n,Tx2n+1)]}{d(Tx2n,Tx2n+1)}$

$+ \gamma [d(Tx2n, Tx2n+1) + d(Tx2n+1, Tx2n+2)]$

$+ \delta . d(Tx2n, Tx2n+1) = (\alpha+\beta+\gamma) d(Tx2n+1, Tx2n+2) + (\gamma+\delta). d(Tx2n, Tx2n+1)$

Therefore, we have $d(Tx_{2n+1}, Tx_{2n+2}) \leq (\alpha+\beta+\gamma) d(Tx2n+1, Tx2n+2) + (\gamma+\delta). d(Tx2n, Tx2n+1)$

Hence, $d(Tx_{2n+1}, Tx_{2n+2}) \leq \frac{(\gamma+\delta)}{1-\alpha-\beta-\gamma} . d(Tx2n, Tx2n+1)$

$= q. d(Tx2n, Tx2n+1)$

Where $\frac{(\gamma+\delta)}{1-\alpha-\beta-\gamma} = q<1$

Sinceα+β+2γ+δ<1.Forn=1,2,3..whetherd(Tx_{2n+1}, Tx_{2n+2})=0 or not similarly we have, d (Tx_{2n+1}, Tx_{2n+2}) ≤ q^n .d(Tx_0, Tx_1)for every positive integer n this means that

$$\sum_{i=0}^{\infty} d(Tx2i+1, Tx2i+2) < \infty \quad \text{…. (b)}$$

Thus the d-completeness of the space implies that the sequence $\{T^n x_0\}$ n ƐN converges to some u in X. So by (a) and (b).$\{E^n x_0\}$ n ƐN and $\{F^n x_0\}$ converges to same point u respectively.

Since E, F and T are continuous, there is a subsequence t of $\{T^n x_0\}$ n ƐN such that E(T(t))→E u ,T(E(t))→T u ,F(T(t))→Fu and T(F(t))→T u Hence by (1.1), we have Eu=Fu=Tu……………………..(c)

Thus,
T(Tu)=T(Eu)=E(Tu)=E(Eu)=E(Fu)=T(Fu)=F(Tu)=F(Eu)=F(Fu)………………………..(d)
Therefore by (1.2), (c) and (d) we have,
If Eu ≠F(Eu)

$$d(Eu, F(Eu)\leq\alpha\frac{d(T(Eu),F(Eu))[1+d(Tu+Eu)]}{1+d(Tu,T(Eu))}$$
$$+\beta.\frac{[d(Tu,Eu).d(T(Eu)F(Eu)]}{d(Tu,T(Eu))}$$
$$+\gamma[d(Tu,Eu)+d(T(Eu),F(Eu))]$$
$$+\delta.d(Tu, T(Eu))$$
$$\leq\delta. d(Tu, T(Eu))$$

Thus we get a contradiction,
Hence E u=F (E u)……………………. (e)
From (d) and (e) we have, E u=F (E u)= T(E u)=E(E u). Hence E u is a common fixed point of E, F, and T. F or uniqueness, Let u, v (u≠ v) is two common fixed points of E, F and T. Then by (1.2) we have d (u, v)=d(E u, F v)

$$\leq \alpha\frac{d(Tv,Fv)[1+d(Tu,Eu)]}{1+d(Tu,Tv)}$$
$$+\beta.\frac{[d(Tv,Ev).d(Tu,Eu)}{d(Tu,Tv)}+\gamma[d(Tv,Fv)+d(Tu,E u)]$$
$$+\delta. d(T u, T v)$$

Therefore, d (u,v) ⩽ δ. d (u,v). Again a contradiction. Therefore u=v Thus T, E, and F have a unique fixed point.

REMARKS:

1. If we put α=γ=0 then we get the result of Pathak and Dubey (6)
2. If we put β=γ=0 then we get the generalized form of the result of Sharma and Agrawal (8).
3. If we put α= β=γ=0 then we get the result of Jungck in L-space.
4. If we put α=γ=0 then we get the generalized form of the result of Jaggi, D.S. in L-space.

COROLARY 1: Let (X, →) be a separated L-space which is d- complete for a non-negative real valued function d on Xx X with d(x, x) = 0 for each x in X. Let E, F and T be three continuous

Self- mappings of X satisfying (1.1) and (1.3)

$$d(E^px,F^qy) \leq\alpha \frac{d(Ty,Fq\, y)[1+d(Tx\, Ep\, x)]}{(1+d(Tx,Ty))}$$
$$+\beta \frac{d(Tx,Epx).d(Ty,Fqy)}{d(Tx,Ty)}$$
$$+\gamma\, [d\, (T\, x,\, E^p x)+Ty,\, F^q y]$$
$$+\delta.\, d\, (Tx,\, Ty) \text{ for all } x, y, \text{ in } X.\ Tx \neq Ty$$

$\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha+\beta+2\gamma+\delta<1$
If some positive integer p, q exist such that E^P, F^q and T are continuous. Then E, F,T have a unique common fixed point in X.

PROOF: **I**t follows from (1.1) $E^p T=TE^P$, $F^q T=T F^q$, $E^q(X)C\ T(X)$ and $F^q(X)C\ T(X)$ by theorem 1,there is a unique fixed point u in X such that,
$u=Tu=E^pu=E^qu$……………………..(f)
That means u is the unique fixed point of T, E^p, F^q .Now T (E u)=E(T u)=E(u)=E(E^pu)=E^p(Eu)…………….(g).
And T (Fu)=F(T u)=F(u)= F(F^q u)=F^q (Fu)…………………………………..(h)
Hence in follows that E u is a common fixed point of T, E^p and Fu is a common fixed point of T and F^q. The uniqueness of u from (f), (g), and (h) implies, u=E u=Fu=T u
This completes the proof.

THEOREM 2: Let (X, →) be a separated L-space which is d-complete for a non-negative real valued function d on Xx X with d(x, x) =0 for each x in X. Let E, F and T be three continuous self- mappings of X satisfying the following conditions.

2.1. EFT=TEF, FET=TEF, EF(X) CT(X) and FF(X) C T(X)

2.2. $d(EFx,FEy)\le\alpha \frac{d(Ty,FEy)[1+d(Tx,EFx)]}{1+d(Tx,Ty)} + \beta.\frac{d(Tx,EFx)d(Ty,FEy)}{d(Tx,Ty)} + \gamma[d(Tx,EFx)+d(Ty,FEy)] + \delta.d(Tx,Ty)$

For all x, y in X where $\alpha, \beta, \gamma, \delta \ge 0$ and $\alpha+\beta+2\gamma+\delta<1$ with $Tx \ne Ty$. Further if T, E F and FE are continuous, Then E, F and T have a common fixed point.

PROOF: Let $EF=S_1$ and $FE=S_2$, then by (2.2),we have

$$d(S_1x,S_2y)\le\alpha\frac{d(Ty,S2y)[1+d(Tx,S1x)]}{1+d(Tx,Ty)} +\beta\frac{[d(Tx,S1x).d(Ty,S2y)]}{d(Tx,Ty)} +\gamma[d(Tx,S1x)+d(Ty,S2y)] +\delta.d(Tx,Ty)$$

Holds for all x, y in X with $Tx \ne Ty$ and $\alpha,\beta,\gamma,\delta \ge 0$ and $\alpha+\beta+2\gamma+\delta \le 1$ and the conditions $S_2T=TS_2$, $S_1T=TS_1$, $S_1(X) C T(X)$ and $S_2(X) C T(X)$ are satisfied. Further by theorem 1 there exists a unique point u such that, $u=S_1u=S_2u=Tu$ Also $Eu=E(S_2u)=EF(Eu)=S_1(Eu)$ and $F(S_1u)=FE(Fu)=S_2(Fu)$ This means that E u is a fixed point of S_1 and Fu is a fixed point of S_2 .The uniqueness of u Implies $u=Eu=Fu=Tu$. This completes the proof**.**

REMARKS: If we put F=1, we get the result of theorem 1.

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